B.Sc. VI SEMESTER

Mathematics

PAPER – I

DIFFERENTIAL EQUATIONS

UNIT-II

Series Solutions Of Ordinary Differential Equations

Syllabus:

Unit – II

Basic definitions, Power series, Ordinary and Singular points, Power series of ODE’s, Frobenius method.

-10HRS

Lecture Notes
By

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Series Solution of ODEs

A powerful and very general technique used to obtain solutions of linear ordinary differential equations (ODEs) is the power series method. One starts by assuming that the solution can be expressed by a simple power series then substitutes the series into the differential equation and thus determines the required values of the coefficients in the power series. In its implementation due to Frobenius the method allows the determination of solutions to a large number of differential equations of great importance in applications.

2.1 Introduction

The linear differential equations we have studied so far all had closed form solutions, that is, their solutions could be expressed in terms of elementary functions, viz. exponential, trigonometric (including inverse trigonometric), polynomial, and logarithmic functions. As we know from calculus courses, most such elementary functions have expansions in terms of power series. Some famous functions with their corresponding power series are:

$$e^x = 1 + x + \frac{x^2}{2!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$cox = 1 - \frac{x^2}{2!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

But there are a whole class of functions, called special functions, which are not elementary functions and which occur frequently in mathematical physics. They usually satisfy second order homogeneous linear differential equations. These equations can sometimes be solved by discovering a power series that satisfies the differential equation but the solution series may not be summable to an elementary function.

2.2.Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... \quad (a_0 \neq 0) \qquad (1)$$

is called a power series in x.

The series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + ... \qquad (2)$$

is a power series in $(x - x_0)$ and is somewhat more general than Eq.(1). However, Eq.(2) can always be reduced to Eq. (1) by replacing $(x - x_0)$ by x.

In this section the most part we shall confine our discussion to power series of the form Eq. (1).

The series Eq. (1) is said to be converge at a point x if the limit i.e. $\lim_{n \to \infty} \left(\sum_{n=0}^{\infty} a_n x^n \right)$ exists and in this case the sum of the series is the value of this limit. It is obvious that (1) always converges at the point $x = 0$.

All the power series in x fall into one or another of three major categories

i.e. $$\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + ... \qquad (3)$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2!} x^2 + ... \qquad (4)$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ... \qquad (5)$$

The series Eq. (3) diverges for $x \neq 0$ and converges for $x = 0$, series (4) converges for all

x and the series Eq. (5) converges for $|x| < 1$ and diverges for $|x| > 1$.

Some power series in x behave like Eq. (3), and converge only for $x = 0$. These are of no interest to us. Some like Eq. (4) converge for all x. These are easiest to work with.

All others are roughly similar to Eq. (5). This means that to each series of this kind these corresponds a positive real number is obtained R, called the **radius of convergence**, with the property that the series converges if $|x| < R$ and diverges for $|x| > R$ ($R = 1$ in the case of Eq. (5)).

It is customary to put R equal to 0 when the series converges only for $x = 0$ and equal to ∞ when it converges for all x. This convention allows us to cover all possibilities in a single statement: each power series in x has a radius of convergence obtained R, where $0 \leq R \leq \infty$ with the property that the series converges if $|x| < R$ and diverges for $|x| > R$. It should be noted that if $R = 0$ then no x satisfies $|x| < R$ and if $R = \infty$ then no x satisfies $|x| > R$

Let $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \dots\dots\dots\dots$ be a series of positive terms.

From elementary calculus, if the $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = l$ is exiats, then by D'Almbert's ratio test the series is converges for $l < 1$ and diverges for $l > 1$.

In the case of the power series Eq. (1) , if for each $a_n \neq 0$ and if for a fixed point $x \neq 0$, we have

$$\lim_{n \to \infty} \left| \frac{a_{n+1}\, x^{n+1}}{a_n\, x^n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1}\, x}{a_n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |x| = l$$

Then the series is converges for $l < 1$ and diverges for $l > 1$.

The power series Eq. (1) converges for $|x| < R$, where $R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$, provided the limit exists.

R is said to be the radius of convergence of power series Eq. (1). The interval $(-R\,,\,R)$ is said to be the interval of convergence.

Since $R=\infty$ for the series Eq. (1), hence the interval of convergence of the power series is $(-\infty\,,\,\infty)$ i.e. the real line.

We shall use the following results

- A power series represents a continuous function within its interval of convergence.
- A power series can be differentiated term wise within its interval of convergence.

Example: Determine the radius of convergence and the interval of convergence of

a) $\sum_{n=0}^{\infty}(n+1)^2 x^n$ b) $\sum_{n=0}^{\infty}\frac{x^n}{3^n}$

.

Solution: a) Here $a_n=(n+1)^2, \quad \therefore a_{n+1}=(n+2)^2$

Now, the radius of convergence i.e. $R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left(\frac{(n+1)^2}{(n+2)^2}\right)$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^2 = \lim_{n\to\infty}\left(\frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)}\right)^2 = \lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)}\right)^2 = \left(\frac{(1+0)}{(1+0)}\right)^2 = 1$$

Therefore, the radius of convergence *i.e.* $R=1$

The interval of convergence is $(-1\,,1)$

b) Here $a_n=\frac{1}{3^n}, \quad \therefore a_{n+1}=\frac{1}{3^{n+1}}$

Now, the radius of convergence i.e.

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left(\frac{3^{n+1}}{3^n}\right) = \lim_{n\to\infty}(3) = 3$$

Therefore, the radius of convergence *i.e.* $R=3$

The interval of convergence is $(-3\,,3)$.

2.3. Analytic function:

A function $f(x)$ defined on an interval containing the point $x = x_0$ is called analytic at x_0 if its Taylor's series,

$\sum_{n=0}^{\infty} \frac{f^n(x)}{n!}(x - x_0)^n$ exists and converges to $f(x)$ for all x in the interval of convergence.

We know that all polynomial functions, exponential functions, trigonometric functions are analytic everywhere.

A rational function is analytic except at those values of x at which its denominator is zero.

For example: The rational function defined $\frac{x}{x^2 - 3x + 2}$ is analytic everywhere except at $x = 1 \ \& \ 2$.

2.4. Series solution of first order ODEs

Example-1: Solve by power series $y' - y = 0$ about $x = 0$.

Solution: Consider the ode $y' - y = 0$

(1)

Now, we find the solution of the above differential equation using series solution.

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots \qquad (2)$$

$$\Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad (3)$$

Using Eq. (2) and (3) in Eq. (1)

i.e. $$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 -$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \{(n+1) a_{n+1} - a_n\} x^n = 0$$

$$\Rightarrow (n+1) a_{n+1} - a_n = 0$$

$$\Rightarrow \ (n+1)a_{n+1} = a_n$$

$$\Rightarrow a_{n+1} = \frac{1}{(n+1)} a_n, \ n = 0,1,2,\ldots\ldots \quad \textbf{(Recurrence relation)}$$

(4)

$$n = 0: \ a_1 = \frac{1}{(0+1)} a_0 = a_0$$

$$n = 1: \ a_2 = \frac{1}{(1+1)} a_1 = \frac{1}{2} a_0 = \frac{1}{2!} a_0 \ ;$$

$$n = 2: \ a_3 = \frac{1}{(2+1)} a_2 = \frac{1}{3} a_2 = \frac{1}{3}\frac{1}{2!} a_0 = \frac{1}{3!} a_0 \ \text{and So on}\ldots$$

Substitute these values in Eq. (2)

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots$$

$$= a_0 + a_0 x + \frac{1}{2!} a_0 x^2 + \frac{1}{3!} a_0 x^3 + \ldots\ldots\ldots$$

$$= a_0 \left(1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots\ldots\ldots \right) = a_0 e^x$$

Therefore $y = a_0 e^x$ is the solution of Eq. (1).

Example-2: Solve by power series at $x = 0$ for $y' + xy = 0$ (1)

Solution: Now, we find the solution of the above differential equation using series solution.

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots \quad (2)$$

$$\Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad (3)$$

Using Eq. (2) and (3) in Eq. (1)

i.e. $$\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = 0 -$$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1) a_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Rightarrow a_1 x^0 + \sum_{n=1}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty}a_{n-1}x^n = 0$$

$$\Rightarrow a_1 + \sum_{n=1}^{\infty}\left\{(n+1)a_{n+1} + a_{n-1}\right\}x^n = 0$$

$$\Rightarrow a_1 = 0 \quad \& \quad (n+1)a_{n+1} + a_{n-1} = 0$$

Now, $(n+1)a_{n+1} + a_{n-1} = 0 \Rightarrow (n+1)a_{n+1} = -a_{n-1}$

$$\Rightarrow a_{n+1} = -\frac{a_{n-1}}{n+1}, \quad n = 1, 2, 3, \ldots\ldots \quad \textbf{(Recurrence relation)} \quad (4)$$

$n = 1: \quad a_2 = -\frac{1}{1+1}a_0 = -\frac{a_0}{2}$;

$n = 2: \quad a_3 = -\frac{1}{2+1}a_1 = -\frac{1}{3}a_1 = 0 \; (\because \; a_1 = 0)$;

$n = 3: \quad a_4 = -\frac{1}{3+1}a_2 = -\frac{1}{4}a_2 = -\frac{1}{4} \times -\frac{1}{2}a_0 = \frac{1}{8}a_0$

$n = 4: \quad a_5 = -\frac{1}{4+1}a_3 = -\frac{1}{5}a_3 = 0 \quad (\because \; a_3 = 0)$

and So on…

Substitute these values in Eq. (2)

$$y = \sum_{n=0}^{\infty}a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots$$

$$= a_0 + (0)x + \left(\frac{a_0}{2}\right)x^2 + (0)\,x^3 + \left(\frac{a_0}{8}\right)x^4 + \ldots\ldots\ldots$$

$$= a_0\left(1 - \left(\frac{x^2}{2}\right) + \frac{1}{2}\left(\frac{x^2}{2}\right)^2 - \ldots\ldots\ldots\right)$$

$$= a_0\left(1 - \left(\frac{x^2}{2}\right) + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2 - \ldots\ldots\ldots\right) = a_0\, e^{-\frac{x^2}{2}}$$

Therefore $y = a_0\, e^{-\frac{x^2}{2}}$ is the solution of Eq. (1).

EXERCISE

I. Solve the following differential equations using power series solution

1. $y' = 2xy \quad \left(\textbf{Ans}: y = a_0 e^{x^2}\right)$

2. $y' + y = 1 \quad \left(\textbf{Ans}: y = 1 + (a_0 - 1)\, e^{-x}\right)$

2.5.Ordinary and singular points:

Consider the second order differential equation of the form,

$$y'' + P(x)y' + Q(x)y = 0 \tag{1}$$

2.6. Ordinary point: A point $x = x_0$ is called an ordinary point of the equation Eq. (1) if both $P(x)$, $Q(x)$ are analytic at $x = x_0$.

2.7. Singular point: If the point $x = x_0$ is not an ordinary point of the equation Eq. (1), then it is called a singular point. There are two types of singular points.

2.8.Regular singular point: A singular point $x = x_0$ is called regular singular point of the equation Eq. (1) if both $(x - x_0)P(x)$, $(x - x_0)^2 Q(x)$ are analytic at $x = x_0$.

2.9.Irregular singular point: A singular point, which is not regular is called irregular singular point.

Example -1: Verify that origin is an ordinary point or regular singular point of the equations

1. $y'' + xy' + y = 0$
2. $2x^2 y'' + xy' - (x+1)y = 0$.

Solution:

1. The equation is $y'' + xy' + y = 0$

   Compare this with $y'' + P(x)y' + Q(x)y = 0$.

   Therefore $P(x) = x$, $Q(x) = 1$

   $\Rightarrow$ At $x = 0$, $P(x)$ & $Q(x)$ are defined.

   $\Rightarrow P(x)$, $Q(x)$ are analytic $x = 0$

   $\Rightarrow$ $x = 0$ is an ordinary point

2. Dividing the equation by $2x^2$ i.e. $y'' + \frac{1}{2x}y' - \frac{(x+1)}{2x^2}y = 0$

Compare this with $y'' + P(x)y' + Q(x)y = 0$.

Therefore $P(x) = \frac{1}{2x}, \quad Q(x) = -\frac{(x+1)}{2x^2}$

$\Rightarrow$ At $x = 0$, $P(x)$ & $Q(x)$ are not defined.

$\Rightarrow P(x)$, $Q(x)$ are not analytic $x = 0$

$\Rightarrow$ $x = 0$ is not an ordinary point

$\Rightarrow$ $x = 0$ is **a singular point**.

Now, $xP(x) = \frac{1}{2}, \quad x^2 Q(x) = -\frac{(x+1)}{2}$

$\Rightarrow x\,P(x)$, $x^2 Q(x)$ are analytic $x = 0$

$\Rightarrow$ $x = 0$ is a **regular singular point**.

Example -2: Verify that $x = 1$ is a regular singular point of the equation

$$(x^2 - 1)y'' + xy' - y = 0.$$

Solution: Dividing the equation by $(x^2 - 1)$ i.e.

$$y'' + \frac{x}{(x^2-1)}y' - \frac{1}{(x^2-1)}y = 0$$

Compare this with $y'' + P(x)y' + Q(x)y = 0$.

Therefore $P(x) = \frac{x}{(x^2-1)}, \quad Q(x) = -\frac{1}{(x^2-1)}$

$\Rightarrow$ At $x = 1$, $P(x)$ & $Q(x)$ are not defined.

$\Rightarrow P(x)$, $Q(x)$ are not analytic $x = 1$

$\Rightarrow$ $x = 1$ is not an ordinary point

$\Rightarrow$ $x = 1$ is **a singular point**.

Now,

$$(x-1)P(x) = (x-1)\frac{x}{(x^2-1)} = \frac{x(x-1)}{(x-1)(x+1)} = \frac{x}{(x+1)} \quad \&$$

$$(x-1)^2 Q(x) = (x-1)^2 \times -\frac{1}{(x^2-1)} = -\frac{(x-1)^2}{(x-1)(x+1)} = -\frac{x-1}{(x+1)}$$

$\Rightarrow$ At $x = 1$, $(x-1)P(x)$ & $(x-1)^2 Q(x)$ are defined.

$\Rightarrow (x-1)P(x)$, $(x-1)^2 Q(x)$ are analytic $x = 1$

$\Rightarrow \quad x = 1$ is a regular singular point.

2.10. Series solution of second order ODEs

a) Series solution about an ordinary point at $x = x_0$:

A point $x = x_0$ is an ordinary point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

if $y,\ y',\ y''$ are regular (i.e. analytic and single-valued) there. The general solution near such an ordinary point can be represented by a Taylor series i.e.

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Example-1: Find the series solution for $y'' + y = 0$ at $x = 0$. (1)

Solution: Comparing the given equation with $y'' + P(x)y' + Q(x)y = 0$

Here $P(x) = 0,\ Q(x) = 1$

Therefore at $x = 0$, $P(x)$, $Q(x)$ are analytic at $x = 0$.

$\Rightarrow x = 0$ is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots$ (2)

be a solution of (1).

$$\therefore y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \& \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Eq. (1) becomes, $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left\{ (n+2)(n+1) a_{n+2} + a_n \right\} x^n = 0$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + a_n = 0$$

$$\Rightarrow a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \ n = 0,1,2,\ldots\ldots.$$

This is the recurrence relation.

$$n = 0: \quad a_2 = -\frac{1}{(0+2)(0+1)}a_0 = -\frac{1}{(2)(1)}a_0 = -\frac{1}{2!}a_0 \;;$$

$$n = 1: \quad a_3 = \frac{1}{(2+1)(1+1)}a_1 = \frac{1}{(3)(2)}a_1 = \frac{1}{3!}a_1 \;;$$

$$n = 2: \quad a_4 = -\frac{1}{(2+2)(2+1)}a_2 = -\frac{1}{(4)(3)}a_2 = -\frac{1}{(4)(3)} \times -\frac{1}{2!}a_0 = \frac{1}{4!}a_0 \;;$$

$$n = 3: \quad a_5 = -\frac{1}{(3+2)(3+1)}a_3 = -\frac{1}{(5)(4)}a_3 = -\frac{1}{(5)(4)} \times -\frac{1}{3!}a_1 = \frac{1}{5!}a_1$$

and So on…

Substitute the values in equation Eq. (2)

i.e. $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots$

$$= a_0 + a_1 x + \left(-\frac{1}{2!}a_0\right)x^2 + \left(-\frac{1}{3!}a_1\right)x^3 + \left(\frac{1}{4!}a_0\right)x^4 + \left(\frac{1}{5!}a_1\right)x^5 + \dots\dots\dots$$

$$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\dots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\dots\right)$$

$$= a_0 \cos x + a_1 \sin x$$

$\therefore\ y = a_0 \cos x + a_1 \sin x$ is the required solution of equation Eq. (1).

Example-2: Find the series solution at the origin

for $(x - 1)y'' + 2y' = 0.$ (1)

Solution: Rewrite the above equation as $y'' + \dfrac{2}{x-1}y' = 0$

Comparing the given equation with $y'' + P(x)y' + Q(x)y = 0$

Here $P(x) = \dfrac{2}{x-1}, \quad Q(x) = 0$

Therefore at $x = 0$, $P(x)$, $Q(x)$ are analytic at $x = 0$.

$\Rightarrow$ $x = 0$ is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots$ (2)

be a solution of Eq. (1).

$$\therefore y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \& \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Eq. (1) becomes, $\quad (x - 1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=1}^{\infty} n\, a_n x^{n-1} = 0$

$$\Rightarrow x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=1}^{\infty} n\, a_n x^{n-1} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2\, n\, a_n x^{n-1} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (n+1)\, n\, a_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x^n + \sum_{n=0}^{\infty} 2\,(n+1)\, a_{n+1} x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (n+1)\, n\, a_{n+1} x^n - \sum_{n=0}^{\infty} \left\{(n+2)(n+1)\, a_{n+2} - 2\,(n+1)\, a_{n+1}\right\} x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (n+1)\, n\, a_{n+1} x^n - \left\{(0+2)(0+1)\, a_2 - 2\,(0+1)\, a_1\right\} x^0 - \sum_{n=1}^{\infty} \left\{(n+2)(n+1)\, a_{n+2} - 2\,(n+1)\, a_{n+1}\right\} x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[(n+1)\, n\, a_{n+1} - \left\{(n+2)(n+1)\, a_{n+2} - 2\,(n+1)\, a_{n+1}\right\}\right] x^n - \left\{(2)(1)\, a_2 - 2\,(1)\, a_1\right\} x^0 = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[(n+1)\, n\, a_{n+1} - (n+2)(n+1)\, a_{n+2} + 2\,(n+1)\, a_{n+1}\right] x^n - \left\{2a_2 - 2a_1\right\} x^0 = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\left\{(n+1)\, n + 2\,(n+1)\right\} a_{n+1} - (n+2)(n+1)\, a_{n+2}\right] x^n - \left\{2a_2 - 2a_1\right\} x^0 = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[(n+1)(n+2) a_{n+1} - (n+2)(n+1)\, a_{n+2}\right] x^n - \left\{2a_2 - 2a_1\right\} x^0 = 0$$

Equating various powers of x

i.e. $\quad 2a_2 - 2a_1 = 0, \quad (n+1)(n+2)a_{n+1} - (n+2)(n+1)\, a_{n+2} = 0$

$$\Rightarrow 2\left(a_2 - a_1\right) = 0, \quad (n+1)(n+2)\left(a_{n+1} - a_{n+2}\right) = 0$$

$$\Rightarrow a_2 - a_1 = 0, \quad a_{n+1} - a_{n+2} = 0$$

$\Rightarrow a_2 = a_1 \;, \quad a_{n+2} = a_{n+1} \;, \; n \geq 1$ (**This is the recurrence relation**)

Since, $\quad a_{n+2} = a_{n+1}$

$n = 1: \quad a_3 = a_2 = a_1 \;(\because a_2 = a_1);$

$n = 2: \quad a_4 = a_3 = a_1 \;(\because a_3 = a_1);$

$n = 3: \quad a_5 = a_4 = a_1 \;(\because a_4 = a_1)$

$n = 4: \quad a_6 = a_5 = a_1 \;(\because a_5 = a_1)$

and So on…

Substitute the values in equation Eq. (2)

i.e. $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots$

$$= a_0 + a_1 x + (a_1)x^2 + (a_1)x^3 + (a_1)x^4 + (a_1)x^5 + \dots\dots\dots$$

$$= a_0 + a_1\left(x + x^2 + x^3 + x^4 + x^5 + \dots\dots\dots\right)$$

$$= a_0 + a_1 x\left(1 + x + x^2 + x^3 + x^4 + \dots\dots\dots\right)$$

$$= a_0 + a_1 \frac{x}{1-x}, \quad \left(\because \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots\dots\dots\right)$$

$\therefore y = a_0 + a_1 \dfrac{x}{1-x}$ is the required solution of equation Eq. (1).

Example-3: Find the series solution for $\left(1+x^2\right)y'' + xy' - y = 0$ at $x = 0$.(1)

Solution: Rewrite the above equation as $y'' + \dfrac{x}{\left(1+x^2\right)}y' - \dfrac{1}{\left(1+x^2\right)}y = 0$

Comparing the given equation with $y'' + P(x)y' + Q(x)y = 0$

Here $P(x) = \dfrac{x}{\left(1+x^2\right)}, \; Q(x) = -\dfrac{1}{\left(1+x^2\right)}$

Therefore at $x = 0$, $P(x)$, $Q(x)$ are analytic at $x = 0$.

$\Rightarrow x = 0$ is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots$ (2)

be a solution of (1).

$$\therefore y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \& \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Eq.(1) becomes, $\left(1+x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n\, a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n\, a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n} + \sum_{n=1}^{\infty} n\, a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n} + \sum_{n=1}^{\infty} n\, a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow (0+2)(0+1)a_{0+2}x^0 + (1+2)(1+1)a_{1+2}x^1 + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$+ \sum_{n=2}^{\infty} n(n-1)a_n x^n + (1)a_1 x^1 + \sum_{n=2}^{\infty} n\, a_n x^n - \left(a_0 x^0 + a_1 x^1 + \sum_{n=2}^{\infty} a_n x^n \right) = 0$$

$$\Rightarrow 2\, a_2 x^0 + (3)(2)a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$$+ a_1 x + \sum_{n=2}^{\infty} n\, a_n x^n - a_0 x^0 - a_1 x - \sum_{n=2}^{\infty} a_n x^n = 0$$

$$\Rightarrow 2\, a_2 x^0 + 6a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$$+ a_1 x + \sum_{n=2}^{\infty} n\, a_n x^n - a_0 x^0 - a_1 x - \sum_{n=2}^{\infty} a_n x^n = 0$$

$$\Rightarrow (2\, a_2 - a_0)x^0 + (6a_3 + a_1 - a_1)x + \sum_{n=2}^{\infty}\left\{(n+2)(n+1)a_{n+2} + n(n-1)a_n + n a_n - a_n\right\}x^n = 0$$

$$\Rightarrow (2\, a_2 - a_0)x^0 + (6a_3)x + \sum_{n=2}^{\infty}\left\{(n+2)(n+1)a_{n+2} + \left[n(n-1) + n - 1\right]a_n\right\}x^n = 0$$

$$\Rightarrow (2\, a_2 - a_0)x^0 + (6a_3)x + \sum_{n=2}^{\infty}\left\{(n+2)(n+1)a_{n+2} + \left[n^2 - n + n - 1\right]a_n\right\}x^n = 0$$

$$\Rightarrow (2\, a_2 - a_0)x^0 + (6a_3)x + \sum_{n=2}^{\infty}\left\{(n+2)(n+1)a_{n+2} + \left[n^2 - 1\right]a_n\right\}x^n = 0$$

Equating various powers of x

i.e. $2\, a_2 - a_0 = 0,\ \ 6a_3 = 0\ ,\ (n+2)(n+1)a_{n+2} + \left[n^2 - 1\right]a_n = 0$

$$\Rightarrow 2\, a_2 = a_0,\ \ a_3 = 0\ ,\ (n+2)(n+1)a_{n+2} = -\left[n^2 - 1\right]a_n$$

$$\Rightarrow a_2 = \frac{a_0}{2},\ \ a_3 = 0\ ,\ a_{n+2} = -\frac{\left[n^2 - 1\right]a_n}{(n+2)(n+1)}\ ,\ n \geq 2$$

Now,

$$a_{n+2} = -\frac{\left[n^2 - 1\right]a_n}{(n+2)(n+1)} = -\frac{(n-1)(n+1)a_n}{(n+2)(n+1)} = -\frac{(n-1)a_n}{(n+2)},\ n = 2, 3, \ldots\ldots ,$$

This is the recurrence relation.

$$n = 2:\ \ a_4 = -\frac{(2-1)a_2}{(2+2)} = -\frac{1}{4}a_2 = -\frac{1}{4}\frac{a_0}{2} = -\frac{a_0}{8}\ \ \left(\because\ a_2 = \frac{a_0}{2} \right);$$

$$n = 3:\ \ a_5 = -\frac{(3-1)a_3}{(3+2)} = -\frac{2}{5}a_3 = 0\ \ (\because\ a_3 = 0\ );$$

$$n = 4:\ \ a_6 = -\frac{(4-1)a_4}{(4+2)} = -\frac{3}{6}a_4 = -\frac{1}{2} \times -\frac{a_0}{8} = \frac{a_0}{16}\ \ \left(\because\ a_4 = -\frac{a_0}{8} \right)$$

and So on…

Substitute the values in equation Eq. (2)

i.e. $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$

$$= a_0 + a_1 x + \left(\frac{1}{2} a_0\right) x^2 + (0) x^3 + \left(-\frac{a_0}{8}\right) x^4 + (0) x^5 + \left(\frac{a_0}{16}\right) x^6 + ...$$

$$= a_0 \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} + ...\right) + a_1 (x)$$

$\therefore y = a_0 \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} + ...\right) + a_1 (x)$ is the required solution of Eq. (1).

EXERCISE

1. Find the power series solution of the equation $y'' + xy' + y = 0$
2. Find the general solution of $(1 + x^2) y'' + 2xy' - 2y = 0$ in terms of power series in x.

$$\left(\begin{array}{l} \textbf{Ans}: 1.\ y = a_0 \left\{ 1 - \frac{x^2}{2} + \frac{x^4}{2.4} - .... \right\} + a_1 \left\{ x - \frac{x^3}{3} + \frac{x^5}{3.5} - .... \right\} \\ \qquad\ 2.\ y = a_0 \left\{ 1 + x^2 - \frac{x^4}{3} - .... \right\} + a_1 x \end{array} \right)$$

b) Series solution about regular singular point at $x = x_0$:

A point $x = x_0$ is an regular singular point of the differential equation

$$y'' + P(x) y' + Q(x) y = 0$$

if y, y', y'' are not all regular there, but $(x - x_0) y', (x - x_0)^2 y$ are all regular at x_0. This essentially implies that $y(x)$ must have a fixed order divergence (or pole) at x_0. The general solution near such a regular singular point can be represented by a **Frobenius series** i.e.

$$y(x) = x^m \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with $a_0 \neq 0$ and note that m is not necessarily an integer.

Example-1: Find the series solution of $4xy'' + 2y' + y = 0$ by Frobenius method.

Solution: Consider the ode $4xy'' + 2y' + y = 0$

(1)

Now, we can rewrite the above Eq. (1) as

$$y'' + \frac{1}{2x} y' + \frac{1}{4x} y = 0$$

Comparing this with $y'' + P(x) y' + Q(x) y = 0$,

Here $P(x) = \frac{1}{2x}$, $Q(x) = \frac{1}{4x}$.

Therefore at $x = 0$, $P(x)$, $Q(x)$ are not analytic at $x = 0$.

$\Rightarrow$ $x = 0$ is not an ordinary point i.e. it is singular point.

Now, $xP(x) = \frac{1}{2}$, $x^2 Q(x) = \frac{x}{4}$ $\Rightarrow$ $xP(x)$, $x^2 Q(x)$ are analytic at $x = 0$.

$\Rightarrow$ $x = 0$ is a regular singular point.

We use **Frobenius method** for the solution of equation Eq. (1).

$$\text{Let } y = x^m \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{m+n} \tag{2}$$

be a solution of Eq. (1).

$$\therefore y' = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1} \quad \& \quad y'' = \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2}.$$

Eq. (1) becomes,

$$4x \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2} + 2 \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 4(m+n)(m+n-1) a_n x^{m+n-1} + \sum_{n=0}^{\infty} 2(m+n) a_n x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \{4(m+n)(m+n-1) + 2(m+n)\} a_n x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2(m+n)\{2(m+n-1)+1\} a_n x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2(m+n)\{2m+2n-2+1\} a_n x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2(m+n)(2m+2n-1) a_n x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2\left(m+n\right)\left(2m+2n-1\right)a_n x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty} 2\left(m+n+1\right)\left(2m+2\left(n+1\right)-1\right)a_{n+1} x^{m+n+1-1} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty} 2\left(m+n+1\right)\left(2m+2\left(n+1\right)-1\right)a_{n+1} x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty} 2\left(m+n+1\right)\left(2m+2n+2-1\right)a_{n+1} x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty} 2\left(m+n+1\right)\left(2m+2n+1\right)a_{n+1} x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow 2\left(m-1+1\right)\left(2m+2\left(-1\right)\right)+1\right)a_0 x^{m-1} + \sum_{n=0}^{\infty} 2\left(m+n+1\right)\left(2m+2n+1\right)a_{n+1} x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow 2\left(m\right)\left(2m-2+1\right)a_0 x^{m-1} + \sum_{n=0}^{\infty}\left\{2\left(m+n+1\right)\left(2m+2n+1\right)a_{n+1} + a_n\right\}x^{m+n} = 0$$

$$\Rightarrow 2\left(m\right)\left(2m-1\right)a_0 x^{m-1} + \sum_{n=0}^{\infty}\left\{2\left(m+n+1\right)\left(2m+2n+1\right)a_{n+1} + a_n\right\}x^{m+n} = 0$$

Equating to zero the coefficient of lowest power and highest power i.e.

$$x^{m-1} : \quad 2\left(m\right)\left(2m-1\right)a_0 = 0 \tag{3}$$

Which is called an **Indicial equation** &

$$x^{m+n} : 2\left(m+n+1\right)\left(2m+2n+1\right)a_{n+1} + a_n = 0 \tag{4}$$

which is called **Recurrence relation**.

Solving Eq. (3) i.e.

$$2\left(m\right)\left(2m-1\right)a_0 = 0 \Rightarrow m\left(2m-1\right) = 0 \quad \left(\because a_0 \neq 0\right)$$

$$\Rightarrow m = 0, \quad m = \frac{1}{2}.$$

Solving Eq. (4) i.e.

$$2\left(m+n+1\right)\left(2m+2n+1\right)a_{n+1} + a_n = 0$$

$$\Rightarrow a_{n+1} = -\frac{1}{2\left(m+n+1\right)\left(2m+2n+1\right)} a_n, \; n \geq 0 \tag{5}$$

When $m = 0$**,** then the equation Eq. (5) reduces to

$$\Rightarrow a_{n+1} = -\frac{1}{2\left(n+1\right)\left(2n+1\right)} a_n, \; n \geq 0$$

Putting the values of $n = 0,1,2,.........$

$$n = 0 : a_1 = -\frac{1}{2\left(1\right)\left(1\right)} a_0 = -\frac{1}{2} a_0 = -\frac{1}{2!} a_0,$$

$$n = 1 : a_2 = -\frac{1}{2\left(1+1\right)\left(2\times 1+1\right)} a_1 = -\frac{1}{\left(4\right)\left(3\right)} a_1 = -\frac{1}{\left(4\right)\left(3\right)} \times -\frac{1}{2!} a_0 = \frac{1}{4!} a_0$$

$n = 2$

$$: a_3 = -\frac{1}{2\,(2+1)(\,2\times 2+1\,)}\, a_2 = -\frac{1}{2\,(3\,)(5)}\, a_2 = -\frac{1}{(6\,)(5)}\times\frac{1}{4!}\, a_0 = -\frac{1}{6!}\, a_0$$

and so on.

Substitute these values in Eq. (2)

$$y = x^m \sum_{n=0}^{\infty} a_n\, x^n = x^m \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots\right)$$

$$y = x^0 \left\{a_0 + \left(-\frac{1}{2!}\, a_0\right) x + \left(\frac{1}{4!}\, a_0\right) x^2 + \left(-\frac{1}{6!}\, a_0\right) x^3 + \ldots\ldots\ldots\right\}$$

$$y = a_0 \left\{1 - \frac{1}{2!}\, x + \frac{1}{4!}\, x^2 - \frac{1}{6!}\, x^3 + \ldots\ldots\ldots\right\}$$

One solution of the given equation is

$$u = 1 - \frac{1}{2!}\, x + \frac{1}{4!}\, x^2 - \frac{1}{6!}\, x^3 + \ldots\ldots\ldots\left(\text{ taking } a_0 = 1\right) \qquad (6)$$

When $m = \frac{1}{2}$**,** then the equation Eq. (5) reduces to

$$\Rightarrow a_{n+1} = -\frac{1}{2\left(\frac{1}{2} + n+1\right)\left(2\times\frac{1}{2} + 2n + 1\right)}\, a_n\ ,\ n \geq 0$$

$$= -\frac{1}{(1+ 2n+2\,)(\,1 + 2\,n + 1\,)}\, a_n = -\frac{1}{(2n+3\,)(2\,n + 2\,)}\, a_n$$

Putting the values of $n = 0,1,2,\ldots\ldots\ldots$

$$n = 0 : a_1 = -\frac{1}{(0+3\,)(0+ 2\,)}\, a_0 = -\frac{1}{(3\,)(2)}\, a_0 = -\frac{1}{3!}\, a_0$$

$$n = 1 : a_2 = -\frac{1}{(2+3\,)(2 + 2\,)} a_1 = -\frac{1}{(5)(4)} a_1 = -\frac{1}{(5)(4)}\times -\frac{1}{3!} a_0 = \frac{1}{5!} a_0$$

$n = 2$

$$: a_3 = -\frac{1}{(2\times 2+3\,)(2\times 2 + 2\,)} a_2 = -\frac{1}{(7)(6\,)} a_2 = -\frac{1}{(7)(6\,)}\times\frac{1}{5!} a_0 = -\frac{1}{7!} a_0$$ and

so on.

Substitute these values in Eq. (2)

$$y = x^m \sum_{n=0}^{\infty} a_n\, x^n = x^m \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots\right)$$

$$y = x^{\frac{1}{2}} \left\{a_0 + \left(-\frac{1}{3!}\, a_0\right) x + \left(\frac{1}{5!}\, a_0\right) x^2 + \left(-\frac{1}{7!}\, a_0\right) x^3 + \ldots\ldots\ldots\right\}$$

$$y = a_0\left(x^{\frac{1}{2}}\left\{1 - \left(\frac{1}{3!}\right)x + \left(\frac{1}{5!}\right)x^2 - \left(\frac{1}{7!}\right)x^3 + \ldots\ldots\ldots\right\}\right)$$

Another solution of the given equation is

$$v = x^{\frac{1}{2}}\left\{1 - \left(\frac{1}{3!}\right)x + \left(\frac{1}{5!}\right)x^2 - \left(\frac{1}{7!}\right)x^3 + \ldots\ldots\ldots\right\} \text{ (taking } a_0 = 1)$$

(7)

Therefore the complete solution of the differential equation is

$y = au + bv$, where u and v are given in equations Eq. (6) and (7) respectively.

Example-2: Solve the differential equation $4xy'' + 2(1 - x)y' - y = 0$ by Frobenius method of power series solution.

Solution: Consider the differential equation $4xy'' + 2(1 - x)y' - y = 0$

(1)

Now, we can rewrite the above Eq. (1) as

$$y'' + \frac{(1 - x)}{2x}y' - \frac{1}{4x}y = 0$$

Comparing this with $y'' + P(x)y' + Q(x)y = 0$

Here $P(x) = \dfrac{(1 - x)}{2x}$, $Q(x) = -\dfrac{1}{4x}$.

Therefore at $x = 0$, $Q(x)$ is not analytic at $x = 0$.

$\Rightarrow$ $x = 0$ is not an ordinary point i.e. it is singular point.

Now, $xP(x) = \dfrac{(1 - x)}{2}$, $x^2 Q(x) = -\dfrac{x}{4}$ $\Rightarrow xP(x), x^2 Q(x)$ are analytic at $x = 0$.

$\Rightarrow$ $x = 0$ is a regular singular point.

We use **Frobenius method** for the solution of equation Eq. (1).

$$\text{Let } y = x^m \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{m+n} \tag{2}$$

be a solution of Eq. (1).

$$\therefore\ y' = \sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} \quad \& \quad y'' = \sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-2}.$$

Eq. (1) becomes,

$$4x\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-2} + 2(1-x)\sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} - \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow 4x\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-2} + 2\sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} - 2x\sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1}$$

$$- \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}4(m+n)(m+n-1)a_n x^{m+n-1} + \sum_{n=0}^{\infty}2(m+n)a_n x^{m+n-1} - 2\sum_{n=0}^{\infty}(m+n)a_n x^{m+n} - \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\{4(m+n)(m+n-1) + 2(m+n)\}a_n x^{m+n-1} - \sum_{n=0}^{\infty}\{2(m+n) + 1\}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}2(m+n)\{2(m+n-1) + 1\}a_n x^{m+n-1} - \sum_{n=0}^{\infty}\{2(m+n) + 1\}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}2(m+n)\{2m + 2n - 2 + 1\}a_n x^{m+n-1} - \sum_{n=0}^{\infty}\{2m + 2n + 1\}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}2(m+n)\{2m + 2n - 1\}a_n x^{m+n-1} - \sum_{n=0}^{\infty}\{2m + 2n + 1\}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty}2(m+n+1)\{2m + 2(n+1) - 1\}a_{n+1} x^{m+n} - \sum_{n=0}^{\infty}\{2m + 2n + 1\}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty}2(m+n+1)\{2m + 2n + 2 - 1\}a_{n+1} x^{m+n} - \sum_{n=0}^{\infty}\{2m + 2n + 1\}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty}2(m+n+1)\{2m + 2n + 1\}a_{n+1} x^{m+n} - \sum_{n=0}^{\infty}\{m + n + 1\}a_n x^{m+n} = 0$$

$$\Rightarrow 2(m + (-1) + 1)\{2m + 2(-1) + 1\}a_0 x^{m-1} + \sum_{n=0}^{\infty}2(m+n+1)\{2m + 2n + 1\}a_{n+1} x^{m+n}$$

$$- \sum_{n=0}^{\infty}\{2m + 2n + 1\}a_n x^{m+n} = 0$$

$$\Rightarrow 2(m - 1 + 1)\{2m - 2 + 1\}a_0 x^{m-1} + \sum_{n=0}^{\infty}2(m+n+1)\{2m + 2n + 1\}a_{n+1} x^{m+n}$$

$$- \sum_{n=0}^{\infty}\{2m + 2n + 1\}a_n x^{m+n} = 0$$

$$\Rightarrow 2(m)\{2m-1\}a_0 x^{m-1} + \sum_{n=0}^{\infty}\left[2(m+n+1)\{2m + 2n + 1\}a_{n+1} - \{2m + 2n + 1\}a_n\right]x^{m+n} = 0$$

$$\Rightarrow 2(m)\{2m-1\}a_0 x^{m-1} + \sum_{n=0}^{\infty}\left[2(m+n+1)\{2m + 2n + 1\}a_{n+1} - \{2m + 2n + 1\}a_n\right]x^{m+n} = 0$$

Equating to zero the coefficient of lowest power and highest power i.e.

$$x^{m-1}: \quad 2(m)\{2m-1\}a_0 = 0 \tag{3}$$

Which is called an **Indicial equation** &

$$x^{m+n}: \quad 2(m+n+1)\{2m+2n+1\}a_{n+1} - \{2m+2n+1\}a_n = 0 \tag{4}$$

which is called **Recurrence relation**.

Solving Eq. (3) i.e.

$$2(m)\{2m-1\}a_0 = 0 \Rightarrow m\{2m-1\} = 0 \quad (\because a_0 \neq 0)$$

$$\Rightarrow \; m = 0 \;\; \& \;\; m = \frac{1}{2}$$

Solving Eq. (4) i.e.

$$2(m+n+1)\{2m+2n+1\}\, a_{n+1} - \{2m+2n+1\} a_n = 0$$

$$\Rightarrow \; (2m+2n+1)\left[2\{m+n+1\}\, a_{n+1} - a_n\right] = 0$$

$$\Rightarrow 2\{m+n+1\}\, a_{n+1} - a_n = 0$$

$$\Rightarrow \; a_{n+1} = \frac{a_n}{2\{m+n+1\}}, \; n \geq 0 \tag{5}$$

When $m = 0$ **,** then the equation Eq. (5) reduces to

$$a_{n+1} = \frac{a_n}{2\{0+n+1\}}, \; n \geq 0$$

$$\Rightarrow \; a_{n+1} = \frac{a_n}{2(n+1)}, \; n \geq 0$$

Putting the values of $n = 0, 1, 2, ...$

$$n = 0: \;\; a_1 = \frac{a_0}{2(0+1)} = \frac{a_0}{2}$$

$$n = 1: \;\; a_2 = \frac{a_1}{2(1+1)} = \frac{a_1}{2(2)} = \frac{a_1}{4} = \frac{1}{4} \times \frac{a_0}{2} = \frac{a_0}{8} \;\; \left(\because a_1 = \frac{a_0}{2}\right)$$

$$n = 2: \;\; a_3 = \frac{a_2}{2(2+1)} = \frac{a_2}{2(3)} = \frac{a_2}{6} = \frac{1}{6} \times \frac{a_0}{8} = \frac{a_0}{48} \;\; \left(\because a_2 = \frac{a_0}{8}\right)$$

and so on.

Substitute these values in Eq. (2)

$$y = x^m \sum_{n=0}^{\infty} a_n x^n = x^m \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...\right)$$

$$y = x^0 \left\{a_0 + \left(\frac{a_0}{2}\right)x + \left(\frac{a_0}{8}\right)x^2 + \left(\frac{a_0}{48}\right)x^3 + ...\right\}$$

$$y = a_0 \left\{1 + \left(\frac{1}{2}\right)x + \left(\frac{1}{8}\right)x^2 + \left(\frac{1}{48}\right)x^3 + ...\right\}$$

One solution of the given equation is

$$u = \left\{1 + \left(\frac{1}{2}\right)x + \left(\frac{1}{8}\right)x^2 + \left(\frac{1}{48}\right)x^3 + ...\right\} \;\; (\text{taking } a_0 = 1) \tag{6}$$

When $m = \frac{1}{2}$**,** then the equation Eq. (5) reduces to

$$a_{n+1} = \frac{a_n}{2\left\{\frac{1}{2} + n + 1\right\}} = \frac{a_n}{2\left\{\frac{1 + 2n + 2}{2}\right\}} = \frac{a_n}{1 + 2n + 2}, \; n \geq 0$$

Putting the values of $n = 0, 1, 2, \ldots\ldots\ldots$

$$n = 0 : a_1 = \frac{a_0}{1 + 2(0) + 2} = \frac{a_0}{1 + 2} = \frac{a_0}{3}$$

$$n = 1 : a_2 = \frac{a_1}{1 + 2(1) + 2} = \frac{a_1}{1 + 2 + 2} = \frac{a_1}{5} = \frac{1}{5}\frac{a_0}{3} = \frac{a_0}{15} \quad \left(\because a_1 = \frac{a_0}{3}\right)$$

$$n = 2 : a_3 = \frac{a_2}{1 + 2(2) + 2} = \frac{a_2}{1 + 4 + 2} = \frac{a_2}{7} = \frac{1}{7}\frac{a_0}{15} = \frac{a_0}{105} \quad \left(\because a_2 = \frac{a_0}{15}\right)$$

and so on.

Substitute these values in Eq. (2)

$$y = x^m \sum_{n=0}^{\infty} a_n x^n = x^m \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots\right)$$

$$y = x^{\frac{1}{2}} \left\{a_0 + \left(\frac{a_0}{3}\right)x + \left(\frac{a_0}{15}\right)x^2 + \left(\frac{a_0}{105}\right)x^2 + \ldots\ldots\ldots\right\}$$

$$y = a_0 \left\{x^{\frac{1}{2}} \left[1 + \left(\frac{1}{3}\right)x + \left(\frac{1}{15}\right)x^2 + \left(\frac{1}{105}\right)x^2 + \ldots\ldots\ldots\right]\right\}$$

Another solution of the given equation is

$$v = x^{\frac{1}{2}} \left[1 + \left(\frac{1}{3}\right)x + \left(\frac{1}{15}\right)x^2 + \left(\frac{1}{105}\right)x^2 + \ldots\ldots\right] (\text{ taking } a_0 = 1)$$

(7)

Therefore the complete solution of the differential equation is

$y = au + bv$, where u and v are given in equations Eq. (6) and (7) respectively.

Example-3: Find the series solution of $2x^2 y'' - xy' + (x - 5)y = 0$. (1)

Solution: Now, we can rewrite the above Eq. (1) as

$$y'' - \frac{1}{2x}y' + \frac{(x - 5)}{2x^2}y = 0$$

Comparing this with $y'' + P(x)y' + Q(x)y = 0$, Here $P(x) = -\frac{1}{2x}$, $Q(x) = \frac{(x-5)}{2x^2}$.

Therefore at $x=0$, $P(x)$, $Q(x)$ are not analytic at $x=0$.

$\Rightarrow$ $x=0$ is not an ordinary point i.e. it is singular point.

Now, $xP(x) = -\frac{1}{2}$, $x^2 Q(x) = \frac{(x-5)}{2}$ $\Rightarrow$ $xP(x)$, $x^2 Q(x)$ are analytic at $x=0$.

$\Rightarrow$ $x=0$ is a regular singular point.

We use **Frobenius method** for the solution of equation Eq. (1).

$$\text{Let } y = x^m \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{m+n} \tag{2}$$

be a solution of Eq. (1).

$\therefore y' = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$ & $y'' = \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2}$.

Eq. (1) becomes,

$$2x^2 \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2} - x \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1} + (x-5) \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow 2x^2 \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2} - x \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1} + x \sum_{n=0}^{\infty} a_n x^{m+n} - 5 \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2(m+n)(m+n-1) a_n x^{m+n} - \sum_{n=0}^{\infty} (m+n) a_n x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n+1} - \sum_{n=0}^{\infty} 5 a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left\{ 2(m+n)(m+n-1) - (m+n) - 5 \right\} a_n x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left\{ (m+n)[2m+2n-3] - 5 \right\} a_n x^{m+n} + \sum_{n=1}^{\infty} a_{n-1} x^{m+n} = 0$$

$$\Rightarrow \left\{ (m+0)[2m+2\times 0-3] - 5 \right\} a_0 x^m + \sum_{n=1}^{\infty} \left\{ (m+n)[2m+2n-3] - 5 \right\} a_n x^{m+n}$$
$$+ \sum_{n=1}^{\infty} a_{n-1} x^{m+n} = 0$$

$$\Rightarrow \left\{ m[2m-3] - 5 \right\} a_0 x^m + \sum_{n=1}^{\infty} \left\{ (m+n)[2m+2n-3] - 5 \right\} a_n x^{m+n}$$
$$+ \sum_{n=1}^{\infty} a_{n-1} x^{m+n} = 0$$

$$\Rightarrow \left\{ 2m^2 - 3m - 5 \right\} a_0 x^m + \sum_{n=1}^{\infty} \left\{ (m+n)[2m+2n-3] - 5 \right\} a_n x^{m+n}$$
$$+ \sum_{n=1}^{\infty} a_{n-1} x^{m+n} = 0$$

$$\Rightarrow \left\{ 2m^2 - 3m - 5 \right\} a_0 x^m + \sum_{n=1}^{\infty} \left(\left\{ (m+n)[2m+2n-3] - 5 \right\} a_n + a_{n-1} \right) x^{m+n} = 0$$

Equating to zero the coefficient of lowest power and highest power i.e.

$$x^m \;:\; \left\{2m^2 - 3m - 5\right\}a_0 = 0 \tag{3}$$

Which is called an **Indicial equation** &

$$x^{m+n} : \; \left\{(m+n)[2m+2n-3] - 5\right\}a_n + a_{n-1} = 0 \tag{4}$$

which is called **Recurrence relation**.

Solving Eq. (3) i.e.

$$\left\{2m^2 - 3m - 5\right\}a_0 = 0 \Rightarrow 2m^2 - 3m - 5 = 0 \qquad (\because a_0 \neq 0)$$

$$\Rightarrow 2m^2 + 2m - 5m - 5 = 0 \Rightarrow 2m(m+1) - 5(m+1) = 0$$

$$\Rightarrow (m+1)(2m-5) = 0 \quad \Rightarrow m = -1 \;\&\; m = \frac{5}{2}$$

Solving Eq. (4) i.e.

$$\left\{(m+n)[2m+2n-3] - 5\right\}a_n + a_{n-1} = 0$$

$$\Rightarrow \left\{(m+n)[2m+2n-3] - 5\right\}a_n = -a_{n-1}$$

$$\Rightarrow a_n = -\frac{a_{n-1}}{\left\{(m+n)[2m+2n-3] - 5\right\}}, \; n \geq 1 \tag{5}$$

When $m = -1$**,** then the equation Eq. (5) reduces to

$$a_n = -\frac{a_{n-1}}{\left\{(-1+n)[-2+2n-3] - 5\right\}}, \; n \geq 1$$

$$\Rightarrow a_n = -\frac{a_{n-1}}{\left\{(n-1)[2n-5] - 5\right\}}, \; n \geq 1$$

Putting the values of $n = 1, 2,$ ....

$$n = 1: \quad a_1 = -\frac{a_0}{\left\{(1-1)[2\times 1 - 5] - 5\right\}} = -\frac{a_0}{\left\{0[2\times 1 - 5] - 5\right\}} = -\frac{a_0}{-5} = \frac{a_0}{5}$$

$$n = 2: \; a_2 = -\frac{a_1}{\left\{(2-1)[2\times 2 - 5] - 5\right\}} = -\frac{a_1}{\left\{1[4-5] - 5\right\}} = -\frac{a_1}{-1-5} = \frac{1}{6}\times\frac{a_0}{5} = \frac{a_0}{30}$$

and so on.

Substitute these values in Eq. (2)

$$y = x^m \sum_{n=0}^{\infty} a_n x^n = x^m\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...\right)$$

$$y = x^{-1}\left\{a_0 + \left(\frac{a_0}{5}\right)x + \left(\frac{a_0}{30}\right)x^2 + ...\right\}$$

$$y = a_0\left(x^{-1}\left\{1 + \left(\frac{1}{5}\right)x + \left(\frac{1}{30}\right)x^2 + ...\right\}\right)$$

One solution of the given equation is

$$u = x^{-1}\left\{1 + \left(\frac{1}{5}\right)x + \left(\frac{1}{30}\right)x^2 + \ldots\ldots\ldots\ldots\right\} \text{ (taking } a_0 = 1)$$

(6)

When $m = \frac{5}{2}$**,** then the equation Eq. (5) reduces to

$$a_n = -\frac{a_{n-1}}{\left\{\left(\frac{5}{2}+n\right)\left[2\left(\frac{5}{2}\right)+2n-3\right]-5\right\}} = -\frac{a_{n-1}}{\left\{\left(\frac{5+2n}{2}\right)[5+2n-3]-5\right\}}$$

$$= -\frac{a_{n-1}}{\left\{\frac{(5+2n)[5+2n-3]-10}{2}\right\}} = -\frac{2a_{n-1}}{(5+2n)[5+2n-3]-10}$$

Putting the values of $n = 1,2,\ldots\ldots\ldots$

$$n = 1: a_1 = -\frac{2a_0}{(5+2\times 1)[5+2\times 1-3]-10} = -\frac{2a_0}{(5+2)[5+2-3]-10} = -\frac{2a_0}{(7)[7-3]-10}$$

$$= -\frac{2a_0}{(7)[7-3]-10} = -\frac{2a_0}{(7)[4]-10} = -\frac{2a_0}{28-10} = -\frac{2a_0}{18} = -\frac{a_0}{9}$$

$$n = 2: a_2 = -\frac{2a_1}{(5+2\times 2)[5+2\times 2-3]-10} = -\frac{2a_1}{(5+4)[5+4-3]-10} = -\frac{2a_1}{(9)[9-3]-10}$$

$$a_2 = -\frac{2a_1}{(9)[6]-10} = -\frac{2a_1}{54-10} = -\frac{2a_1}{44} = -\frac{a_1}{22} = -\frac{1}{22}\times -\frac{a_0}{9} = \frac{a_0}{198}\left(\because a_1 = -\frac{a_0}{9}\right)$$

and so on.

Substitute these values in Eq. (2)

$$y = x^m \sum_{n=0}^{\infty} a_n x^n = x^m\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots\ldots\right)$$

$$y = x^{\frac{5}{2}}\left\{a_0 + \left(-\frac{1}{9}a_0\right)x + \left(\frac{1}{198}a_0\right)x^2 + \ldots\ldots\ldots\ldots\right\}$$

$$y = a_0\left(x^{\frac{5}{2}}\left\{1 - \left(\frac{1}{9}\right)x + \left(\frac{1}{198}\right)x^2 + \ldots\ldots\ldots\ldots\right\}\right)$$

Another solution of the given equation is

$$v = x^{\frac{5}{2}}\left\{1 - \left(\frac{1}{9}\right)x + \left(\frac{1}{198}\right)x^2 + \ldots\ldots\ldots\ldots\right\} \text{ (taking } a_0 = 1)$$

(7)

Therefore the complete solution of the differential equation is

$y = au + bv$, where u and v are given in equations Eq. (6) and (7) respectively.

EXERCISE 2.1

Find the power series solution of the following differential equations by Frobenius method

1. $2xy'' + (3 - x)y' - y = 0$
2. $2x^2y'' + xy' - (x + 1)y = 0$

$$\left(\begin{array}{l} \textbf{Ans}: 1.\ u = 1 + \dfrac{x}{1.3} + \dfrac{x^2}{1.\,3.5} + \dots\dots\dots,\ v = x^{-\frac{1}{2}}\left\{1 + x + \dfrac{x^2}{2^2\,2!} + \dots\dots\dots\right\} \\ \qquad\ \ 2.\ u = x\left\{1 + \dfrac{x}{5} + \dfrac{x^2}{70} + \dots\dots\dots\right\},\ v = x^{-\frac{1}{2}}\left\{1 - x - \dfrac{x^2}{2} + \dots\dots\dots\right\} \end{array} \right)$$

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